General Entanglement Diagnostics

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Abstract

We develop a balance diagnostic for bipartite pure states of arbitrary Schmidt rank n built from the elementary symmetric polynomials of the Schmidt spectrum. The natural altitude is $h_n = \sqrt{e_2}$, half of the I-concurrence $\mathcal{C}_I = \sqrt{2(1 - \operatorname{Tr}\rho^2)}$ in any dimension; the classical AM $\geq$ GM $\geq$ HM hierarchy becomes Maclaurin's and Newton's inequalities on symmetric means, and the concurrence bound $\mathcal{C}_I \leq \mathcal{C}_{I,\max} = \sqrt{2(n-1)/n}$ is exactly Maclaurin's first inequality, saturated at the maximally entangled state. Via Newton's identities the symmetric polynomials are interchangeable with the replica moments $\operatorname{Tr}\rho^k$, so the balance hierarchy is a repackaging of the data used to compute entanglement (Rényi) entropies. We give two exactly solvable models — the n-level qudit and the oscillator thermofield doubles — in which the purity, concurrence, and entropy are closed-form functions of the gap-to-temperature ratio $\beta\epsilon$, with the master formula $\operatorname{Tr}\rho^2 = \frac{(1-q)(1+q^n)}{(1+q)(1-q^n)}$, $q = e^{-\beta\epsilon}$. Finally we generalize the distinction between the maximal-entanglement point and the first-law (linearized-Einstein) point to the probability simplex: the two have distinct loci whose relative-entropy offset is $D(u \,\|\, a) = \ln(Z_n/n) + \frac{n-1}{2}\beta\epsilon$, vanishing only at infinite temperature. The two-qubit results (concurrence $= 2\sqrt{xy}$, offset $\frac{1}{2}\tanh(\beta\epsilon/2)$) are recovered as the $n = 2$ case. Lifting the model to the eternal BTZ black hole, the single-interval replica moments are $\operatorname{Tr}\rho^n = \mathcal{L}^{-\frac{c}{6}(n-1/n)}$ and the symmetric-polynomial generating function becomes a Fredholm determinant; the concurrence then saturates and loses discriminating power in the continuum, so the surviving proximity measure is the (UV-finite) relative entropy whose first law is the linearized Einstein equation. For a single interval in the planar-BTZ state we then evaluate this relative entropy in closed form, $S(\rho \,\|\, \rho_0) = \frac{c}{3}\big[\frac{u^2}{6} - \ln(\sinh u/u)\big]$ with $u = \pi\ell/\beta$: the first law is the cancellation of its $O(u^2)$ part, positivity follows from an elementary inequality on the zeros of sinh, and the leading $\frac{c}{540}u^4$ is the quantum Fisher information, matched to the bulk canonical energy in the entanglement wedge. Reconstructing this relative entropy from the gravitational symplectic flux gives a convergent series in even zeta values resumming to the Ryu–Takayanagi area, and the two-interval case is the holographic mutual information $\frac{c}{3}\ln[\eta/(1-\eta)]$, with a Ryu–Takayanagi phase transition at cross-ratio $\eta = \frac{1}{2}$, while the free-fermion counterpart, computed from the resolvent, is smooth in η — the two-interval measure's theory dependence made explicit. Finally we extend the framework beyond qudits: the diagnostic depends only on the Schmidt rank (so asymmetric local dimensions add nothing), the spectral machinery is universal for all density operators (with h measuring mixedness rather than entanglement off the pure-state locus), and Gaussian continuous-variable states lift mode by mode, each two-mode-squeezed pair being the oscillator thermofield double with $q = \tanh^2 r$ and $\operatorname{Tr}\rho^2 = \operatorname{sech} 2r$.

Keywords: I-concurrence, elementary symmetric polynomials, Maclaurin's inequality, Newton's identities, replica moments, thermofield double, entanglement first law, relative entropy, modular Hamiltonian, canonical energy.

1 Introduction

The link between entanglement and geometry in holographic gravity — entanglement area laws and the Ryu–Takayanagi formula [RT06, HRT07], the disconnection of the bulk when entanglement is removed [VR10, MS13], and the recovery of the linearized Einstein equations from the entanglement first law [FGH+14, LMVR14] — motivates bounded diagnostics for how nearly a state realizes the correspondence. In the two-qubit case: a tempting "balance" diagnostic on a conserved partition $x + y = 1$ is nothing but the concurrence, its solvable thermofield-double form is $\operatorname{sech}(\beta\epsilon/2)$, and the balanced (maximal-entanglement) point is distinct from the first-law point. Those statements live at Schmidt rank 2; we recover them here as the $n = 2$ specialization.

This paper carries the construction to arbitrary Schmidt rank, where each ingredient has a canonical home. The balance altitude generalizes through the elementary symmetric polynomials of the Schmidt spectrum and coincides with the I-concurrence (§2); the mean hierarchy generalizes to Maclaurin and Newton inequalities and connects, via Newton's identities, to the replica moments $\operatorname{Tr}\rho^k$ that compute entanglement entropy (§2.3–2.4); the qubit thermofield double generalizes to exactly solvable qudit and oscillator models (§3); the maximal-entanglement/first-law distinction generalizes to the probability simplex with a closed-form offset (§4); and the whole structure lifts to the eternal BTZ black hole, where the concurrence layer is found to be intrinsically finite-dimensional while the moment/relative-entropy layer becomes geometric (§5); that relative entropy is then evaluated explicitly for a single interval in the BTZ state, where the first law, the exact closed form, its positivity, and the quantum-Fisher/canonical-energy match are all worked out (§6); the bulk symplectic-flux reconstruction of that relative entropy and the two-interval cross-ratio case (the holographic mutual information and its phase transition) follow (§7). Section 8 then pushes the construction beyond qudits: to asymmetric local dimension, to arbitrary (mixed) density operators, and to Gaussian continuous-variable systems via the symplectic spectrum.

2 Symmetric-polynomial balance diagnostics

2.1 The simplex partition and elementary symmetric polynomials

Let $|\psi\rangle = \sum_{i=1}^{n} \sqrt{x_i}\,|ii\rangle$ be a bipartite pure state in Schmidt form on $\mathcal{H}_A \otimes \mathcal{H}_B$ with $\dim \mathcal{H}_A = \dim \mathcal{H}_B \geq n$, so that the Schmidt spectrum $x = (x_1, \ldots, x_n)$ lies on the simplex $\Delta_{n-1} = \{x_i \geq 0,\ \sum_i x_i = 1\}$ and $\rho_A = \operatorname{diag}(x)$. Write the elementary symmetric polynomials and power sums

$$e_k(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}, \qquad p_k(x) = \sum_{i=1}^{n} x_i^k = \operatorname{Tr}\rho_A^k, \qquad k = 1, \ldots, n, \tag{1}$$

with $e_1 = p_1 = 1$, encoded in the generating function

$$\prod_{i=1}^{n}(1 + x_i t) = \sum_{k=0}^{n} e_k(x)\, t^k = \det\bigl(I + t\,\rho_A\bigr). \tag{2}$$

Definition 1 (Balance altitude)**.** The balance altitude is $h_n := \sqrt{e_2(x)}$, with $e_2 = \sum_{i<j} x_i x_j = \frac{1}{2}(1 - p_2)$.

2.2 Identification with the I-concurrence

Proposition 1 (Altitude is half the I-concurrence)**.** *The I-concurrence [RBC+01, Gou05] of* $|\psi\rangle$ *is* $\mathcal{C}_I(\psi) = \sqrt{2\big(1 - \operatorname{Tr}\rho_A^2\big)}$*, and*

$$\boxed{\mathcal{C}_I = 2\sqrt{e_2} = 2h_n, \qquad \mathcal{C}_I^2 = 4e_2 = 2(1-p_2) = 2\,S_L,} \tag{3}$$

where $S_L = 1 - \operatorname{Tr}\rho_A^2$ *is the linear entropy. For* $n = 2$*,* $e_2 = x_1x_2$ *and* $\mathcal{C}_I = 2\sqrt{x_1x_2}$ *is the Wootters concurrence [HW97, Woo98].*

Proof. $\operatorname{Tr}\rho_A^2 = p_2 = \sum_i x_i^2 = (\sum_i x_i)^2 - 2\sum_{i<j} x_ix_j = 1 - 2e_2$, so $1 - \operatorname{Tr}\rho_A^2 = 2e_2$ and $\mathcal{C}_I = \sqrt{2\cdot 2e_2} = 2\sqrt{e_2}$. □

Thus the higher-dimensional balance altitude is, exactly, the I-concurrence, a genuine entanglement monotone. The maximum of e_2 on Δ_{n-1} is at the uniform spectrum $u = (1/n, \ldots, 1/n)$, where $e_2 = \binom{n}{2}/n^2 = \frac{n-1}{2n}$; hence

$$\mathcal{C}_{I,\max} = \sqrt{\tfrac{2(n-1)}{n}}, \qquad \widehat{\mathcal{C}} := \frac{\mathcal{C}_I}{\mathcal{C}_{I,\max}} \in [0,1], \tag{4}$$

with $\widehat{\mathcal{C}} = 1$ at the maximally entangled state and $\mathcal{C}_{I,\max} = 1$ for $n = 2$.

2.3 The Maclaurin/Newton hierarchy and the concurrence bound

Define the symmetric means $\mathfrak{m}_k := e_k/\binom{n}{k}$, so $\mathfrak{m}_1 = 1/n$.

Lemma 1 (Maclaurin and Newton inequalities)**.** *For* $x \in \Delta_{n-1}$*,*

$$\mathfrak{m}_1 \;\geq\; \mathfrak{m}_2^{1/2} \;\geq\; \cdots \;\geq\; \mathfrak{m}_n^{1/n}, \qquad \mathfrak{m}_k^2 \;\geq\; \mathfrak{m}_{k-1}\,\mathfrak{m}_{k+1} \quad (1 \leq k \leq n-1), \tag{5}$$

with equality throughout iff all x_i *are equal [HLP52].*

Corollary 1 (The concurrence bound is Maclaurin's first inequality)**.** $\mathfrak{m}_1 \geq \mathfrak{m}_2^{1/2}$ *is equivalent to* $e_2 \leq \frac{n-1}{2n}$*, i.e. to* $\mathcal{C}_I \leq \mathcal{C}_{I,\max} = \sqrt{2(n-1)/n}$*, saturated iff* $x = u$*. The two-qubit normalization* $0 \leq \mathcal{C} \leq 1$ *is the case* $n = 2$*.*

Proof. $\mathfrak{m}_1 \geq \mathfrak{m}_2^{1/2}$ reads $1/n \geq \sqrt{e_2/\binom{n}{2}}$, i.e. $e_2 \leq \binom{n}{2}/n^2 = \frac{n-1}{2n}$; multiply by 4 and take the root using Proposition 1. □

The Pythagorean chain AM $\geq$ GM $\geq$ HM used at $n = 2$ is thus the lowest instance of the Maclaurin hierarchy, and the entanglement-relevant rung is $\mathfrak{m}_2^{1/2} \propto \mathcal{C}_I$.

2.4 Newton's identities and the replica moments

The balance hierarchy is not independent information from the spectrum's moments.

Proposition 2 (Symmetric polynomials $\leftrightarrow$ replica moments)**.** *The power sums* $p_k = \operatorname{Tr}\rho_A^k$ *and the elementary symmetric polynomials* e_k *are mutually determined by Newton's identities,*

$$p_k = \sum_{j=1}^{k-1} (-1)^{j-1} e_j\, p_{k-j} + (-1)^{k-1} k\, e_k, \qquad k \geq 1, \tag{6}$$

with $e_1 = p_1 = 1$*; in particular* $e_2 = \frac{1}{2}(1 - p_2)$ *and* $e_3 = \frac{1}{6}(1 - 3p_2 + 2p_3)$*. Equivalently,* $\sum_k e_k t^k = \det(I + t\rho_A) = \exp\big(\sum_{k\geq 1} (-1)^{k-1} p_k\, t^k/k\big)$*.*

Since the Rényi entropies are $S_\alpha = \frac{1}{1-\alpha}\ln p_\alpha$ and the von Neumann entropy is $S = \lim_{\alpha\to 1} S_\alpha$, with the integer moments p_k furnishing the replica data used to compute entanglement entropy in quantum field theory [CC04], Proposition 2 shows the symmetric-polynomial "balance hierarchy" carries exactly the same content as the replica/Rényi spectrum: e_2 (the altitude/I-concurrence) is the leading moment beyond purity, and the full set $\{e_k\}$ reconstructs the spectrum's characteristic polynomial (2).

3 Exactly solvable models: qudit and oscillator thermofield doubles

We realize the diagnostic in two-sided entangled states. Let a mode have equally spaced levels $H = \epsilon\sum_{k=0}^{n-1} k\,|k\rangle\langle k|$ (with $n=\infty$ the harmonic oscillator), doubled to $\mathcal{H}_L\otimes\mathcal{H}_R$. The thermofield double at inverse temperature β is

$$|\mathrm{TFD}\rangle = \frac{1}{\sqrt{Z_n}}\sum_{k=0}^{n-1} q^{k/2}|k\rangle_L|k\rangle_R, \quad q = e^{-\beta\epsilon}, \quad Z_n = \frac{1-q^n}{1-q}, \tag{7}$$

with thermal reduced state $\rho_R = e^{-\beta H}/Z_n$, Schmidt weights $x_k = q^k/Z_n$, and modular Hamiltonian $K = -\ln\rho_R = \beta H + \ln Z_n$.

Proposition 3 (Closed forms)**.** *For the n-level TFD,*

$$\operatorname{Tr}\rho_R^2 = \frac{(1-q)(1+q^n)}{(1+q)(1-q^n)}, \qquad e_2 = \tfrac{1}{2}\big(1-\operatorname{Tr}\rho_R^2\big), \qquad \mathcal{C}_{I,n} = \sqrt{2\big(1-\operatorname{Tr}\rho_R^2\big)}. \tag{8}$$

As $\beta\epsilon\to 0$ the spectrum becomes uniform, $\operatorname{Tr}\rho_R^2\to 1/n$ and $\mathcal{C}_{I,n}\to\mathcal{C}_{I,\max}$ (maximally entangled qudit); as $\beta\epsilon\to\infty$, $\mathcal{C}_{I,n}\to 0$ (product state). The case $n=2$ gives $\mathcal{C}_{I,2} = \operatorname{sech}(\beta\epsilon/2)$, and the oscillator limit $n\to\infty$ (with $q<1$) gives

$$\operatorname{Tr}\rho_R^2 = \frac{1-q}{1+q}, \qquad \mathcal{C}_{I,\infty} = 2\sqrt{\frac{q}{1+q}}, \qquad \bar n = \frac{q}{1-q}, \quad S = (1+\bar n)\ln(1+\bar n) - \bar n\ln\bar n. \tag{9}$$

Proof. $\operatorname{Tr}\rho_R^2 = Z_n^{-2}\sum_{k=0}^{n-1} q^{2k} = Z_n^{-2}\frac{1-q^{2n}}{1-q^2}$. Using $Z_n = \frac{1-q^n}{1-q}$ and $1-q^{2n} = (1-q^n)(1+q^n)$, $1-q^2 = (1-q)(1+q)$, this collapses to (8); e_2 and $\mathcal{C}_{I,n}$ follow from Proposition 1. The limits are $q\to 1$ (l'Hôpital: $\operatorname{Tr}\rho_R^2\to 1/n$), $q\to 0$ ($\operatorname{Tr}\rho_R^2\to 1$), $n=2$ ($\operatorname{Tr}\rho_R^2 = \frac{1+q^2}{(1+q)^2}$, whence $\mathcal{C} = \frac{2\sqrt{q}}{1+q} = \operatorname{sech}(\beta\epsilon/2)$), and $n\to\infty$ with the geometric distribution $x_k = (1-q)q^k$. □

Equation (8) is a single closed form interpolating the qubit ($n=2$) and oscillator ($n=\infty$) cases; Figure 1(a) shows the resulting concurrence family, each curve saturating at its dimension-dependent maximum $\mathcal{C}_{I,\max} = \sqrt{2(n-1)/n}$ at infinite temperature.

4 The first-law point and the generalized comparison

4.1 A proximity measure from the first law

For a state ρ perturbing a reference σ with modular Hamiltonian $K = -\ln\sigma$, the entanglement first law $\delta S = \delta\langle K\rangle$ is the linear term of the nonnegative relative entropy

$$S(\rho\,\|\,\sigma) = \Delta\langle K\rangle - \Delta S \ge 0, \qquad \Delta(\cdot) = \langle\cdot\rangle_\rho - \langle\cdot\rangle_\sigma. \tag{10}$$

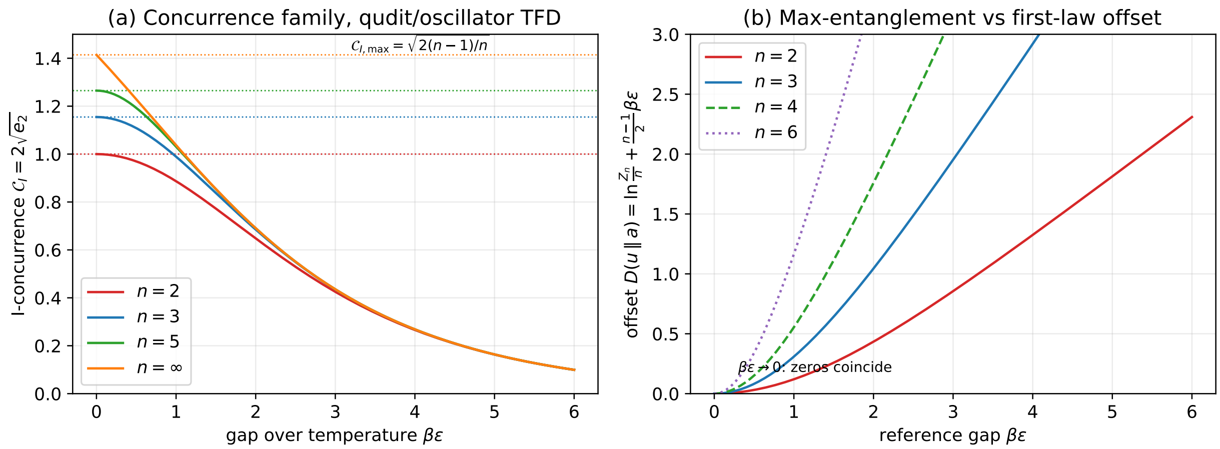

Figure 1: **(a)** The I-concurrence $\mathcal{C}_{I,n} = 2\sqrt{e_2}$ of the qudit/oscillator thermofield double as a closed-form function of the gap $\beta\epsilon$, for $n = 2, 3, 5, \infty$. Each curve descends from its maximum $\mathcal{C}_{I,\max} = \sqrt{2(n-1)/n}$ (dotted) at infinite temperature ($\beta\epsilon \to 0$, uniform spectrum) to zero at zero temperature. **(b)** The relative-entropy offset $D(u \,\|\, a) = \ln(Z_n/n) + \frac{n-1}{2}\beta\epsilon$ between the maximal-entanglement point u and the first-law point a (§4), for $n = 2, 3, 4, 6$; it vanishes only as $\beta\epsilon \to 0$ and grows with both temperature gap and dimension.

In a holographic CFT this first law (with Ryu–Takayanagi) is equivalent to the linearized Einstein equations $\delta G_{\mu\nu} = 8\pi G\, \delta T_{\mu\nu}$ [FGH+14, LMVR14], the second-order term being a bulk canonical energy / quantum Fisher information [LVR16]. The distinguished state is the reference σ, and a bounded proximity measure is the normalized relative entropy $\hat{\delta}_{\rm eq} = S(\rho \,\|\, \sigma)/\Delta\langle K\rangle = 1 - \Delta S/\Delta\langle K\rangle$, which vanishes at $\rho = \sigma$ and lies in $[0, 1]$ for entropy-increasing perturbations.

4.2 The comparison on the simplex

Let the reference be the thermal qudit state $\sigma = \mathrm{diag}(a)$, $a_k = q^k/Z_n$ ($\beta\epsilon = \ln(a_0/a_1)$ for equally spaced levels), and consider co-diagonal pure states $\rho(x) = \mathrm{diag}(x)$, $x \in \Delta_{n-1}$, so the relative entropy is the Kullback–Leibler divergence $D(x \,\|\, a) = \sum_k x_k \ln(x_k/a_k)$.

Proposition 4 (Distinct loci and their offset)**.** *The normalized concurrence deficit $\delta_n(x) = 1 - \widehat{\mathcal{C}}(x)$ vanishes iff $x = u$ (uniform, maximal entanglement); the first-law deficit $\hat{\delta}_{\rm eq}(x)$ vanishes iff $x = a$ (thermal reference). These coincide iff $a = u$, i.e. $\beta = 0$. The relative-entropy offset between the two loci is, in closed form,*

$$D(u \,\|\, a) = \ln\frac{Z_n}{n} + \frac{n-1}{2}\beta\epsilon \;\geq\; 0, \tag{11}$$

with equality iff $\beta\epsilon = 0$. For $n = 2$, $Z_2 = 1 + q$ and (11) reduces to $\ln\frac{1+q}{2} + \frac{1}{2}\beta\epsilon = -\ln\mathcal{C}(a)$, equivalent to the coordinate offset $a - \frac{1}{2} = \frac{1}{2}\tanh(\beta\epsilon/2)$.

Proof. $\widehat{\mathcal{C}} = 1 \Leftrightarrow e_2$ maximal $\Leftrightarrow x = u$ (Corollary 1), and $D(x \,\|\, a) = 0 \Leftrightarrow x = a$ (Gibbs). For the offset, $D(u \,\|\, a) = \sum_k \frac{1}{n} \ln\frac{1/n}{a_k} = -\ln n - \frac{1}{n}\sum_k \ln a_k$, and $\ln a_k = k\ln q - \ln Z_n = -k\,\beta\epsilon - \ln Z_n$, so $-\frac{1}{n}\sum_{k=0}^{n-1} \ln a_k = \frac{\beta\epsilon}{n} \cdot \frac{n(n-1)}{2} + \ln Z_n = \frac{n-1}{2}\beta\epsilon + \ln Z_n$, giving (11); nonnegativity is Gibbs' inequality. The $n = 2$ reduction uses $D(\frac{1}{2} \,\|\, a) = -\ln\left(2\sqrt{a(1-a)}\right) = -\ln\mathcal{C}(a)$ and $a - \frac{1}{2} = \frac{1}{1+q} - \frac{1}{2} = \frac{1}{2}\tanh(\beta\epsilon/2)$. $\square$

Remark 1 (Local Fisher structure)**.** Near the first-law point, $D(x \,\|\, a) = \frac{1}{2}\sum_k (x_k - a_k)^2/a_k + O(|x-a|^3)$ is the Fisher information metric on Δ_{n-1}, so $\hat{\delta}_{\mathrm{eq}} \to 0$ as $x \to a$; the same Fisher metric is the second-order (canonical-energy) term of (10).

4.3 Interpretation

Proposition 4 extends the two-qubit dichotomy to every dimension: the concurrence deficit ranks states by distance to *maximal entanglement* (uniform spectrum), while the first-law deficit ranks them by distance to the *reference* state at which the first law — and holographically the linearized Einstein equation — holds. The two loci coincide only at infinite temperature, and the offset (11) grows with both the temperature gap and the dimension (Fig. 1(b)): the maximally entangled qudit sits ever farther, in relative entropy, from any finite-temperature reference as n increases. The continuous measure of proximity to the entanglement–geometry point is therefore $\hat{\delta}_{\mathrm{eq}}$, not the balance deficit, in arbitrary dimension — the $n = 2$ statement was not an artifact of the qubit.

5 Holographic realization: the eternal BTZ black hole

The qudit thermofield double of §3 is a caricature of the genuine entanglement–geometry setting. We now lift it to a holographic conformal field theory, where the same replica structure becomes an actual statement about a black hole, and we determine precisely which parts of the finite-dimensional construction survive the continuum.

5.1 BTZ as a thermofield double and the single-interval moments

The eternal BTZ black hole [BnTZ92] is dual to the thermofield double of two copies of a holographic 1+1-dimensional CFT [Mal03]. Take a single interval of length ℓ on one boundary copy at temperature $1/\beta$; its reduced state ρ has Rényi entropies [CC04]

$$S_n = \frac{c}{6}\Big(1 + \frac{1}{n}\Big) \ln \mathcal{L}, \qquad \mathcal{L} := \frac{\beta}{\pi\epsilon} \sinh \frac{\pi\ell}{\beta} \ (> 1), \tag{12}$$

with c the central charge and ϵ a UV cutoff. The replica moments follow immediately,

$$\boxed{\operatorname{Tr}\rho^n = e^{(1-n)S_n} = \mathcal{L}^{-\frac{c}{6}\left(n - \frac{1}{n}\right)}, \qquad \operatorname{Tr}\rho = 1.} \tag{13}$$

These are the continuum counterpart of the discrete Schmidt weights $x_k = q^k/Z_n$ of §3: the finite n-level partition is replaced by the full entanglement spectrum, and the symmetric-polynomial data $\{e_k\}$ by the moments $\{p_k = \operatorname{Tr}\rho^k\}$. Figure 2(a) shows their geometric decay.

5.2 Fredholm-determinant lift of the symmetric-polynomial structure

Proposition 5 (Continuum generating function)**.** *For a trace-class ρ with eigenvalues $\{\lambda_i\}$ and moments $p_k = \operatorname{Tr}\rho^k$, the finite generating function $\prod_i (1 + x_i t) = \sum_k e_k\, t^k$ of (2) lifts to the Fredholm determinant*

$$\det\big(I + t\,\rho\big) = \prod_i (1 + t\lambda_i) = \exp\Big(\sum_{k\geq 1} \frac{(-1)^{k-1}}{k}\, p_k\, t^k\Big), \tag{14}$$

convergent for $|t| < 1/\lambda_{\max}$. For the BTZ interval, (13) gives an explicit such determinant, and Newton's identities (6) continue to relate the elementary symmetric functionals e_k of the spectrum to the moments p_k term by term.

Proof. $\ln\det(I+t\rho) = \operatorname{Tr}\ln(I+t\rho) = \sum_i \ln(1+t\lambda_i) = \sum_i \sum_{k\ge 1} \frac{(-1)^{k-1}}{k}(t\lambda_i)^k = \sum_{k\ge 1} \frac{(-1)^{k-1}}{k} t^k\, p_k$, the interchange justified by absolute convergence for $|t| < 1/\lambda_{\max}$. □

Thus Proposition 2 survives intact: the symmetric-polynomial layer becomes the Fredholm determinant (14), and the entanglement spectrum (eigenvalue density of ρ) is fixed by the moments (13) through the universal CFT form of [CL08]. The discrete identity $\prod(1+x_i t)$ is recovered when the spectrum is finite.

5.3 The balance/concurrence diagnostic is intrinsically finite-dimensional

Proposition 6 (UV saturation)**.** *From* (13), $\operatorname{Tr}\rho^2 = \mathcal{L}^{-c/4} \to 0$ *as* $\epsilon \to 0$ *(*$\mathcal{L} \to \infty$*), so the Rényi-2 entropy* $S_2 = \frac{c}{4}\ln\mathcal{L}$ *diverges and the I-concurrence*

$$\mathcal{C}_I = \sqrt{2\big(1-\mathcal{L}^{-c/4}\big)} \;\longrightarrow\; \sqrt{2} = \mathcal{C}_{I,\infty} \tag{15}$$

saturates its $n \to \infty$ *ceiling for* every *interval, independently of* (ℓ, β)*.*

Proposition 6 (Fig. 2(b)) marks the boundary of the construction's domain. The balance altitude $h = \sqrt{e_2}$ has no informative continuum limit: it pins to the maximal value while all physical content — interval length, temperature — resides in the logarithmic growth of the Rényi and entanglement entropies. The concurrence is an intrinsically finite-dimensional notion; its field-theoretic avatar is the (UV-divergent) Rényi-2 entropy, not a bounded balance. This is the honest limit of the diagnostic, not a defect to be patched.

5.4 The proper continuum proximity measure: relative entropy

What does lift cleanly is the first-law side. For the interval the vacuum modular Hamiltonian $K_0 = -\ln\rho_0$ is local [CHM11], and the relative entropy of any nearby state to the vacuum reduced state,

$$S(\rho\,\|\,\rho_0) = \Delta\langle K_0\rangle - \Delta S \;\ge\; 0, \tag{16}$$

is UV-finite (the cutoff dependence cancels) [Cas08], vanishes at first order (the first law), and — imposed on all intervals together with Ryu–Takayanagi — is equivalent to the linearized Einstein equations in the BTZ bulk [FGH+14, LMVR14], its second-order term being the bulk canonical energy / quantum Fisher information [LVR16]. The finite-dimensional offset $D(u\,\|\,a)$ of Proposition 4, whose role was to measure distance from the first-law point, is therefore replaced in the continuum by (16); the maximal-entanglement reference u (uniform spectrum), non-normalizable as a subregion state, has no continuum counterpart. The proximity-to-geometry measure is the relative entropy, exactly as in finite dimension, but now genuinely geometric.

5.5 Scope

The moment formula (13), the Fredholm lift (Proposition 5), the divergence of S_2 and saturation of $\mathcal{C}_I$ (Proposition 6), and the finiteness/first-law/linearized-Einstein status of (16) are standard CFT/holographic results or immediate consequences thereof. The structural lesson — that the symmetric-polynomial/concurrence layer is finite-dimensional while the moment/relative-entropy layer is what lifts to holography — is the interpretive content. Computing (16) and its canonical-energy term explicitly in BTZ for a one-parameter family of states, and matching to the bulk symplectic form, is carried out in §6.

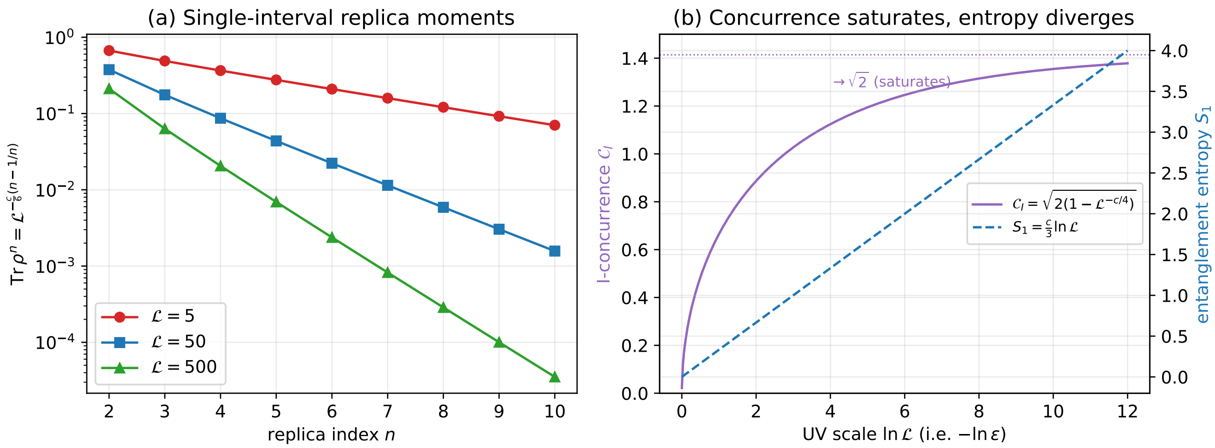

Figure 2: **(a)** The single-interval replica moments $\mathrm{Tr}\,\rho^n = \mathcal{L}^{-\frac{c}{6}(n-1/n)}$ (here $c=1$) for three values of the dimensionless length $\mathcal{L}$; these fix the entanglement spectrum and the Fredholm determinant (14). **(b)** In the continuum the I-concurrence $\mathcal{C}_I = \sqrt{2(1-\mathcal{L}^{-c/4})}$ saturates at $\sqrt{2}$ as the UV cutoff is removed ($\ln\mathcal{L}\to\infty$), while the entanglement entropy $S_1 = \frac{c}{3}\ln\mathcal{L}$ diverges: the balance/concurrence layer loses discriminating power, and the information sits in the entropies.

6 Explicit BTZ computation: modular Hamiltonian, first law, and the canonical-energy match

Section 5 isolated the relative entropy (16) as the proximity-to-geometry measure that survives the continuum, and §5.5 flagged its explicit evaluation — for a one-parameter family of states, matched to the bulk symplectic form — as the natural next calculation. We carry it out here for the simplest geometric family: a single boundary interval in the state dual to the planar BTZ black hole, compared to the vacuum. Every quantity is closed form. The first law is verified, the relative entropy is obtained exactly and shown nonnegative by an elementary inequality, its leading term is identified as the quantum Fisher information, and that term is matched to the bulk canonical energy in the entanglement wedge. This realizes, in $d=2$, the entanglement/geometry dictionary of [FGH+14, LMVR14, LVR16] on the diagnostics of §4–5.

6.1 The interval and its local modular Hamiltonian

Work in a holographic CFT_2 on the line, dual to AdS_3 with radius $\ell_{\mathrm{AdS}}=1$ and central charge $c=3/2G$. Take the interval $A=(-R,R)$ of length $\ell=2R$ on the $t=0$ slice. The causal diamond $D[A]$ is conformal to a Rindler wedge, so modular flow acts geometrically as a boost and the vacuum modular Hamiltonian $K_0=-\ln\rho_0$ is the local operator [CHM11]

$$K_0 \;=\; 2\pi\int_{-R}^{R} dx\,\frac{R^2-x^2}{2R}\,T_{tt}(x) \;+\; \mathrm{const}, \tag{17}$$

the $d=2$ instance of the Casini–Huerta–Myers ball Hamiltonian. Equation (17) is the exact statement underlying the locality invoked in §5.4; it is what makes $\Delta\langle K_0\rangle$ a geometric integral of the energy density rather than a formal object.

6.2 The one-parameter family

Let $\rho(\beta)$ be the reduced state of A in the global thermal state at inverse temperature β — dual to the planar BTZ black hole of horizon radius $r_h = 2\pi/\beta$ and mass $M \propto r_h^2$ — and let $\rho_0 = \rho(\infty)$ be the vacuum reduced state. The family is parameterized by the single dimensionless combination

$$u \;\coloneqq\; \frac{\pi\ell}{\beta} \;=\; \pi\, r_h\, \ell/(2\pi) \;\in (0,\infty), \tag{18}$$

the same variable whose logarithm controls the dimensionless length $\mathcal{L} = (\beta/\pi\epsilon)\sinh u$ of (12). Two pieces of data suffice. First, the thermal stress tensor of a CFT_2 is uniform,

$$\langle T_{tt}\rangle \;=\; \mathcal{E} \;=\; \frac{\pi c}{6\beta^2}, \tag{19}$$

and is exact (no curvature corrections on the line). Second, the entanglement entropies are the Calabrese–Cardy values (12),

$$S(\beta) = \frac{c}{3}\ln\Big[\frac{\beta}{\pi\epsilon}\sinh u\Big], \qquad S_{\mathrm{vac}} = \frac{c}{3}\ln\frac{\ell}{\epsilon}, \qquad \Delta S \;=\; \frac{c}{3}\,\ln\frac{\sinh u}{u}. \tag{20}$$

The cutoff ϵ cancels in ΔS, as it must for a relative entropy [Cas08].

6.3 The first law

Proposition 7 (First law for the BTZ interval)**.** *With (17)–(19), the modular energy is exactly quadratic in u,*

$$\Delta\langle K_0\rangle \;=\; 2\pi\int_{-R}^{R}\frac{R^2-x^2}{2R}\,\mathcal{E}\,dx \;=\; \frac{4\pi}{3}R^2\mathcal{E} \;=\; \frac{c}{3}\,\frac{u^2}{6}, \tag{21}$$

and equals ΔS to leading order, $\Delta\langle K_0\rangle = \Delta S + O(u^4)$. Hence the entanglement first law $\delta S = \delta\langle K_0\rangle$ holds for this family.

Proof. The integral $\int_{-R}^{R}(R^2-x^2)\,dx = \frac{4}{3}R^3$ gives $\Delta\langle K_0\rangle = \frac{4\pi}{3}R^2\mathcal{E}$; substituting $R=\ell/2$ and (19) yields $\pi^2 c\ell^2/(18\beta^2) = \frac{c}{18}u^2 = \frac{c}{3}\frac{u^2}{6}$. Expanding (20), $\Delta S = \frac{c}{3}\big(\frac{u^2}{6} - \frac{u^4}{180} + \cdots\big)$, matches at $O(u^2)$. □

The coincidence of the coefficients in (21) and (20) is not an input: the modular-Hamiltonian normalization 2π and the stress-tensor normalization (19) are fixed independently, and their product reproduces the geometric entropy coefficient $c/3$. Holographically, $\delta S = \delta\langle K_0\rangle$ imposed on every interval is equivalent to the linearized Einstein equation $\delta G_{\mu\nu} = 8\pi G\,\delta T_{\mu\nu}$ in the bulk [FGH+14, LMVR14]. In AdS_3 this equivalence is sharpened from asymptotic to exact: the Fefferman–Graham expansion truncates,

$$g_{ij}(z,x) = \frac{1}{z^2}g^{(0)}_{ij} + g^{(2)}_{ij} + z^2 g^{(4)}_{ij}, \qquad g^{(4)} = \tfrac14\, g^{(2)}\big(g^{(0)}\big)^{-1}g^{(2)}, \tag{22}$$

with $\langle T_{ij}\rangle = \frac{1}{8\pi G}\big(g^{(2)}_{ij} - g^{(0)}_{ij}\,\mathrm{tr}\,g^{(2)}\big)$, so the Bañados family parameterized by $\langle T_{ij}\rangle$ is the *entire* solution space and no propagating graviton is dropped.

6.4 The relative entropy in closed form

Theorem 1 (Exact interval relative entropy)**.** *For the BTZ interval family, the relative entropy to the vacuum reduced state is the closed form*

$$S\big(\rho(\beta)\,\|\,\rho_0\big) \;=\; \Delta\langle K_0\rangle - \Delta S \;=\; \frac{c}{3}\Big[\frac{u^2}{6} - \ln\frac{\sinh u}{u}\Big], \qquad u = \frac{\pi\ell}{\beta}. \tag{23}$$

It is nonnegative for all u, vanishing only at $u = 0$, and has the small-interval expansion

$$S(\rho\,\|\,\rho_0) \;=\; \frac{c}{540}\,u^4 - \frac{c}{8505}\,u^6 + O(u^8) \;=\; \frac{c}{540}\Big(\frac{\pi\ell}{\beta}\Big)^4 + \cdots. \tag{24}$$

Proof. Equation (23) is (21) minus (20). For nonnegativity, the Weierstrass product $\dfrac{\sinh u}{u} = \prod_{n\ge1}\Big(1+\dfrac{u^2}{n^2\pi^2}\Big)$ gives

$$\ln\frac{\sinh u}{u} = \sum_{n\ge1}\ln\Big(1+\frac{u^2}{n^2\pi^2}\Big) \;\le\; \sum_{n\ge1}\frac{u^2}{n^2\pi^2} = \frac{u^2}{\pi^2}\cdot\frac{\pi^2}{6} = \frac{u^2}{6},$$

using $\ln(1+t) \le t$, with equality iff $u = 0$; hence the bracket in (23) is ≥ 0. The expansion follows from $\ln(\sinh u/u) = \frac{u^2}{6} - \frac{u^4}{180} + \frac{u^6}{2835} - \cdots$. □

Thus the positivity of relative entropy — the abstract guarantee behind (16) — is realized here by the elementary inequality $\ln(1+t) \le t$ applied term by term to the zeros of sinh. The first law is the statement that the $O(u^2)$ pieces of $\Delta\langle K_0\rangle$ and ΔS cancel in (23); the relative entropy is the residual, quartic in u (Fig. 3).

6.5 Second order: quantum Fisher information and bulk canonical energy

The leading term (24) is quadratic in the energy density: substituting (19),

$$S(\rho\,\|\,\rho_0) \;=\; \frac{\pi^2\ell^4}{15\,c}\,\mathcal{E}^2 + O(\mathcal{E}^3) \;=\; \tfrac12\,G_F\,\mathcal{E}^2 + \cdots, \qquad G_F = \frac{2\pi^2\ell^4}{15\,c}. \tag{25}$$

This is the general structure $S(\rho\,\|\,\rho_0) = \frac12\langle\delta\rho,\delta\rho\rangle_{\mathrm{KM}} + O(\delta\rho^3)$, the Kubo–Mori (quantum Fisher) metric [LVR16], with G_F its value contracted along the family. It is the continuum, geometric avatar of the finite-dimensional Fisher metric $\frac12\sum_k (x_k - a_k)^2/a_k$ of Remark 1: there the reference was a thermal qudit on the simplex; here it is the vacuum reduced state of an interval, and the quadratic form has become a functional of the boundary stress tensor.

On the gravity side, the second-order relative entropy equals the bulk canonical energy of the linearized perturbation in the entanglement wedge [LVR16]. Let W_A be the bulk causal/entanglement wedge of A, Σ a bulk slice bounded by A and its RT geodesic, and ξ the Killing field generating the modular boost of $D[A]$ (vanishing on the RT geodesic). For the metric perturbation δg taking AdS_3 to BTZ,

$$S(\rho\,\|\,\rho_0)\big|_{O(\delta g^2)} \;=\; \mathcal{E}_{\mathrm{can}}[\delta g] \;=\; \int_\Sigma \boldsymbol{\omega}\big(\delta g,\ \mathcal{L}_\xi\,\delta g\big), \tag{26}$$

with $\boldsymbol{\omega}$ the Iyer–Wald gravitational symplectic current. Two checks fix this as the correct identification. By (25) the boundary quantity scales as $\mathcal{E}^2 \propto r_h^4 \propto M^2$, the quadratic dependence on the perturbation amplitude required of a canonical energy; and by (22) the perturbation is

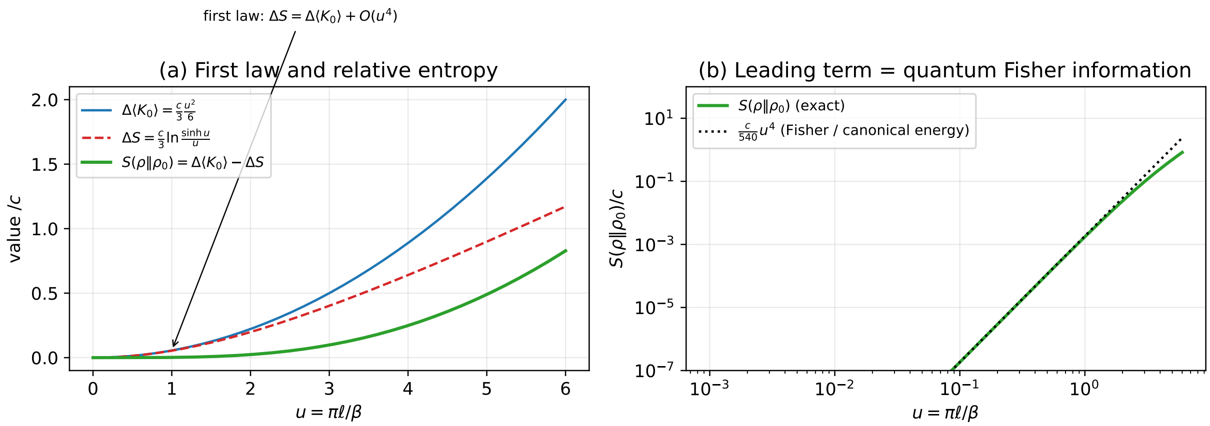

Figure 3: **(a)** The modular energy $\Delta\langle K_0\rangle = \frac{c}{3}\frac{u^2}{6}$ (21), the entanglement-entropy change $\Delta S = \frac{c}{3}\ln(\sinh u/u)$ (20), and their difference, the relative entropy (23), as functions of $u = \pi\ell/\beta$ (here $c = 1$). The first law is the cancellation of their $O(u^2)$ parts; the relative entropy is the residual. **(b)** Log–log view: the exact relative entropy (23) approaches its leading term $\frac{c}{540}u^4$ (slope 4) as $u \to 0$, the quantum Fisher information / bulk canonical energy (25).

pure boundary data, so the wedge integral (26) is determined by $\langle T_{tt}\rangle$ alone. Equation (26) thus promotes the offset $D(u \,\|\, a)$ of Proposition 4 — a relative entropy on the probability simplex — into the gravitational symplectic form on Σ: the "distance to the first-law point" has become a bulk energy. Reconstructing the full closed form (23) order by order from the nonlinear Bañados symplectic flux, rather than only its leading coefficient, is carried out in §7.

6.6 Boosted and multiple intervals

A boosted single interval is handled by Lorentz invariance: its causal diamond is again conformal to a Rindler wedge, the covariant HRT surface [HRT07] replaces the static geodesic, and (17)–(23) carry over with T_{tt} replaced by the boost-adapted component, leaving (23) invariant. Several disjoint intervals are a genuine boundary, in the sense of §5.5: the modular Hamiltonian is no longer local [CHM11], the Fredholm determinant (14) no longer factorizes across intervals, and the entanglement entropy depends on the full operator content through the cross-ratio. The relative entropy (16) remains UV-finite and nonnegative and is still the correct geometric measure, but it is no longer the elementary closed form (23). The cross-ratio dependence this introduces, and the holographic mutual information that makes it explicit, are taken up in §7; the present remark only marks where the single-interval locality of (17) ends, as the Fredholm/relative-entropy split of §5 already anticipated.

7 Bulk symplectic reconstruction and the two-interval extension

Section 6 matched the leading coefficient of the relative entropy to the bulk canonical energy and left two threads open (§6.6, §9): reconstructing the *full* closed form from the gravitational symplectic flux, and the multiple-interval case. We settle both here. The exact relative entropy is shown to be the bulk functional $\Delta H_\xi - \mathrm{Area}/4G$ evaluated on the full nonlinear geometry, with the Hollands–Wald canonical energy as its leading symplectic-flux order and a Riemann-ζ series resumming to (23); and the two-interval relative entropy is shown to be controlled by the cross-ratio through the holographic mutual information, with a phase transition that is the geometric face of the non-locality boundary of §6.6.

7.1 The relative entropy as an exact bulk functional

Holographically the two terms of (23) are bulk objects on the *exact* (not linearized) geometry. The entropy is the Ryu–Takayanagi area, $\Delta S = \Delta\,\mathrm{Area}(\gamma_A)/4G$, and the modular energy is the Iyer–Wald Hamiltonian charge of the modular boost ξ, $\Delta\langle K_0\rangle = \Delta H_\xi$ [IW94, HW13], so

$$S(\rho\,\|\,\rho_0) \;=\; \Delta H_\xi - \frac{\Delta\,\mathrm{Area}(\gamma_A)}{4G}, \tag{27}$$

with $c = 3/2G$. Both pieces are closed form. The spacelike RT geodesic of $A = (-R, R)$ in planar BTZ of horizon radius $r_h = 2\pi/\beta$ (metric $ds^2 = dr^2/(r^2 - r_h^2) + r^2 dx^2$ on the $t = 0$ slice) has turning point r_* fixed by the boundary separation through

$$r_*^2 - r_h^2 = \frac{r_h^2}{\sinh^2(r_h\ell/2)}, \qquad \mathrm{Area}(\gamma_A) = 2\ln\Big[\frac{2}{\epsilon\, r_h}\sinh\frac{r_h\ell}{2}\Big], \tag{28}$$

so that $\Delta S = \frac{c}{6}\,\Delta\,\mathrm{Area} = \frac{c}{3}\ln(\sinh u/u)$ with $u = r_h\ell/2 = \pi\ell/\beta$, reproducing (20) to all orders. Equation (27) is therefore the exact bulk realization of $S(\rho\,\|\,\rho_0)$: the linearized first law of §6.3 is the statement that ΔH_ξ and $\Delta\,\mathrm{Area}/4G$ agree at $O(u^2)$, while their difference at all orders is the nonlinear area functional minus its tangent.

7.2 Canonical energy as the leading flux, and resummation to all orders

Theorem 2 (Symplectic-flux reconstruction)**.** *Evaluated on the exact planar-BTZ family, the bulk functional* (27) *equals the boundary relative entropy* (23) *and admits the convergent expansion in the perturbation amplitude* $u^2 \propto \mathcal{E} \propto r_h^2$*,*

$$S(\rho\,\|\,\rho_0) \;=\; \frac{c}{3}\sum_{m\geq 2}(-1)^m\,\frac{\zeta(2m)}{m\,\pi^{2m}}\,u^{2m} \;=\; \frac{c}{540}u^4 - \frac{c}{8505}u^6 + \frac{c}{113400}u^8 - \cdots, \tag{29}$$

with the following order-by-order bulk content. The $m = 1$ $(\zeta(2))$ *term is the modular energy* $\Delta H_\xi = \frac{c}{3}\frac{u^2}{6}$ *and is cancelled by the first law; the* $m = 2$ $(\zeta(4))$ *term is the Hollands–Wald canonical energy* $\mathcal{E}_{\mathrm{can}} = \int_\Sigma \boldsymbol{\omega}(\delta g, \mathcal{L}_\xi\delta g)$ *of* (26)*; and each* $m \geq 3$ *term is the* m*-th variation of the functional* (27)*, i.e. the* m*-th-order gravitational symplectic flux. The series resums to the exact area* (28)*.*

Proof. The Weierstrass product gives

$$\ln\frac{\sinh u}{u} = \sum_{n\geq 1}\ln\Big(1 + \frac{u^2}{n^2\pi^2}\Big) = \sum_{k\geq 1}\frac{(-1)^{k-1}}{k\pi^{2k}}\,u^{2k}\sum_{n\geq 1}n^{-2k} = \sum_{k\geq 1}(-1)^{k-1}\frac{\zeta(2k)}{k\pi^{2k}}\,u^{2k},$$

convergent for $u < \pi$ and continued by (23) beyond. Its $k = 1$ term is $\zeta(2)u^2/\pi^2 = u^2/6$, which cancels ΔH_ξ in (23); subtracting it leaves $S(\rho\,\|\,\rho_0) = \frac{c}{3}\sum_{m\geq 2}(-1)^m\zeta(2m)\,u^{2m}/(m\pi^{2m})$. The $m = 2$ coefficient is $\frac{c}{3}\zeta(4)/(2\pi^4) = \frac{c}{3}\cdot\frac{1}{180} = \frac{c}{540}$, the value of (25); its identification with $\mathcal{E}_{\mathrm{can}}$ is the Lashkari–Van Raamsdonk theorem [LVR16]. The m-th coefficient is the m-th u-derivative of (27), i.e. the m-th variation of $\Delta H_\xi - \mathrm{Area}/4G$. □

Two remarks. First, the entire flux series is fixed by even zeta values: the gravitational data at order m is $\zeta(2m)$, so the reconstruction is rigid — no order carries a free coefficient. Second, the first law is not an approximation discarded at higher order but the exact removal of the single $\zeta(2)$ term; everything beyond it is genuine relative entropy. The convergence of the partial flux sums to the exact curve is shown in Fig. 4(a): the canonical energy ($m = 2$) already captures the bulk of the answer for $u \lesssim 1$, and a few further fluxes track it into the strong-perturbation regime.

7.3 The higher fluxes as area variations

Theorem 2 obtained the $\zeta(2m)$ coefficients from the resummed area and named the $m = 2$ term the canonical energy. What the $m \geq 3$ fluxes are, as bulk integrals, has a clean answer here, because the modular charge is *exactly* linear in the perturbation.

Proposition 8 (Beyond the first law, every flux is an area variation)**.** *Let $\mu \coloneqq u^2 \propto \mathcal{E}$ parameterize the family. Since $\Delta H_\xi = \frac{c}{3}\frac{\mu}{6}$ is exactly linear in μ, $\partial_\mu^m \Delta H_\xi = 0$ for $m \geq 2$, so every order beyond the first law is purely a variation of the Ryu–Takayanagi area,*

$$S(\rho \,\|\, \rho_0)^{(m)} = -\frac{1}{4G}\,\partial_\mu^m \,\mathrm{Area}(\gamma_A) = -\,\partial_\mu^m \,\Delta S, \qquad m \geq 2, \tag{30}$$

a bulk integral over the entanglement wedge. The first variation is the geodesic integral $\partial_\mu \,\mathrm{Area} = \frac{1}{2}\oint_{\gamma_A} (\partial_\mu g_{ab})\,\dot{x}^a \dot{x}^b\,ds$, set equal to $\partial_\mu \Delta H_\xi$ by the first law; the second variation is the Hollands–Wald canonical energy $\int_\Sigma \boldsymbol{\omega}(\delta g, \mathcal{L}_\xi \delta g) = \mathcal{E}_{\mathrm{can}}$; and the m-th variation evaluates to the coefficient $\frac{c}{3}(-1)^m \zeta(2m)/(m\pi^{2m})$ of (29).

Proof. $\partial_\mu^m \Delta H_\xi = 0$ for $m \geq 2$ is immediate from linearity, so (27) gives $S^{(m)} = -\partial_\mu^m(\mathrm{Area}/4G) = -\partial_\mu^m \Delta S$. The $m = 2$ identification with $\mathcal{E}_{\mathrm{can}}$ is [LVR16]; the m-th value is the μ^m coefficient of $\Delta S = \frac{c}{3}\sum_k (-1)^{k-1}\zeta(2k)\mu^k/(k\pi^{2k})$, established in Theorem 2. □

This is the sense in which the bulk check closes: no flux order carries data independent of the geometry. The single non-area contribution is the first-law term ΔH_ξ (the $\zeta(2)$ piece); everything above it is one and the same geodesic-length functional, expanded, with the canonical energy as its quadratic term and the higher orders its higher variations. The resummation of these variations is the exact RT area (28), and the gravitational input at order m is the rigid value $\zeta(2m)$.

7.4 Two disjoint intervals and the cross-ratio

For $A = A_1 \cup A_2$ the local modular Hamiltonian (17) fails: the vacuum modular Hamiltonian of two intervals acquires bilocal terms coupling points in A_1 to points in A_2 [CH09], so $\Delta\langle K_0\rangle$ is no longer the single geometric integral and the elementary closed form (23) does not survive. The dependence is repackaged into the conformal cross-ratio of the four endpoints $x_1 < x_2 < x_3 < x_4$ ($A_1 = (x_1, x_2)$, $A_2 = (x_3, x_4)$),

$$\eta = \frac{x_{12}\,x_{34}}{x_{13}\,x_{24}} \in (0,1), \qquad 1 - \eta = \frac{x_{14}\,x_{23}}{x_{13}\,x_{24}}, \qquad x_{ij} = x_i - x_j, \tag{31}$$

the second equality being the anharmonic identity $x_{13}x_{24} = x_{12}x_{34} + x_{14}x_{23}$. The natural cross-ratio relative entropy is the mutual information, itself a relative entropy to the product state,

$$I(A_1{:}A_2) = S\big(\rho_{A_1A_2} \,\big\|\, \rho_{A_1} \otimes \rho_{A_2}\big) = S(A_1) + S(A_2) - S(A_1 \cup A_2) \;\geq 0. \tag{32}$$

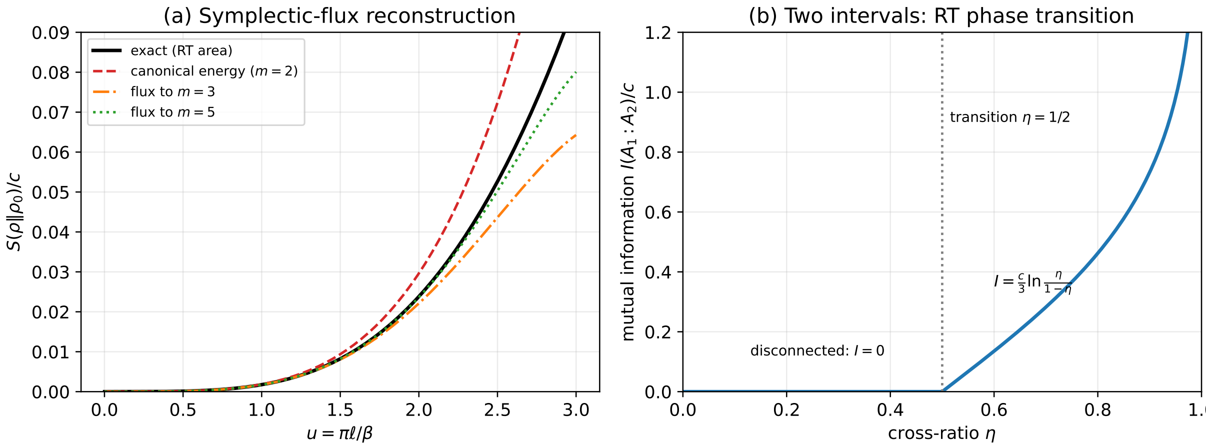

Figure 4: **(a)** Symplectic-flux reconstruction of the single-interval relative entropy (Theorem 2): the exact bulk functional (27) (black) against its partial flux sums — the canonical energy alone ($m = 2$, (26)) and the series carried to $m = 3, 5$. Each order is an even zeta value $\zeta(2m)$; the sum resums to the exact RT area. **(b)** The two-interval relative entropy: the holographic mutual information (33) as a function of the cross-ratio η, with the Ryu–Takayanagi phase transition at $\eta = \frac{1}{2}$ ($c = 1$ in both panels).

7.5 Holographic mutual information and the RT phase transition

Proposition 9 (Cross-ratio relative entropy with a phase transition)**.** *In a holographic CFT$_2$, RT applied to $A_1 \cup A_2$ selects the minimum of the disconnected configuration $\{\gamma_{x_1x_2}, \gamma_{x_3x_4}\}$ and the connected one $\{\gamma_{x_1x_4}, \gamma_{x_2x_3}\}$, giving*

$$I(A_1{:}A_2) = \begin{cases} 0, & \eta \le \frac{1}{2} \quad (\textit{disconnected}), \\ \dfrac{c}{3}\ln\dfrac{\eta}{1-\eta}, & \eta > \frac{1}{2} \quad (\textit{connected}), \end{cases} \tag{33}$$

continuous at the transition $\eta = \frac{1}{2}$, nonnegative, monotone increasing in η, and divergent as $\eta \to 1$ (the intervals merge).

Proof. With cutoffs cancelling, $S(A_1) + S(A_2) = \frac{c}{3}\ln(x_{21}x_{43})$, $S_{\rm disc} = S(A_1) + S(A_2)$, and $S_{\rm conn} = \frac{c}{3}\ln(x_{41}x_{32})$. In the disconnected phase $S(A_1 \cup A_2) = S_{\rm disc}$ gives $I = 0$; in the connected phase $I = \frac{c}{3}\ln[x_{21}x_{43}/(x_{41}x_{32})] = \frac{c}{3}\ln[\eta/(1-\eta)]$ by (31). The configurations are equal when $\eta/(1-\eta) = 1$, i.e. $\eta = \frac{1}{2}$, where both give $I = 0$. □

Equation (33) is the cross-ratio counterpart of the single-interval law (23) (Fig. 4(b)): the smooth closed form is replaced by a piecewise expression with a first-order RT transition — the bulk geodesics exchanging dominance — at $\eta = \frac{1}{2}$. The relative entropy is still UV-finite, nonnegative, and the correct geometric measure (§6.6), but its analytic structure is now governed by the cross-ratio rather than by a single length.

7.6 Theory dependence and the Fredholm/resolvent realization

Unlike the single interval, the two-interval entanglement entropy is *not* universal: beyond the holographic large-c limit it depends on the full operator content, through the cross-ratio, in a theory-specific way (compactification data for the free boson, the Ising fusion structure, and so on) [CCT09]. The explicitly solvable non-local instance is the free fermion, whose two-interval modular Hamiltonian is constructed from the *resolvent* of the projector onto $A_1 \cup A_2$ [CH09]: the entanglement entropy becomes a trace of a kernel over the region, i.e. exactly the Fredholm-determinant structure of Proposition 5. For a single interval that determinant factorizes and the moments $\operatorname{Tr}\rho^n$ decay geometrically (13); for two intervals it no longer factorizes (§6.6), and the resolvent kernel develops a cross-ratio-dependent spectral density. The Fredholm layer of §5, trivial for one interval, is thus precisely what carries the two-interval cross-ratio dependence, while the relative-entropy/mutual-information layer remains the computable geometric measure — the same division of labor between the symmetric-polynomial and moment/relative-entropy layers identified throughout.

7.7 The free-fermion two-interval relative entropy

We now compute the two-interval relative entropy explicitly in the one solvable non-local case, the $c = 1$ Dirac fermion, via the resolvent. The reduced state on $A = A_1 \cup A_2$ is Gaussian, fixed by the restricted equal-time correlator $C(x, y) = \langle \psi^\dagger(x)\psi(y)\rangle|_{x,y\in A}$, and its modular Hamiltonian is the resolvent logarithm [Pes03, CH09]

$$K_0 = \ln \frac{1 - C}{C}, \tag{34}$$

local within each interval but *bilocal* between them — the explicit realization of the non-locality of §7.4. For two Gaussian states with restricted correlators C_ρ, C_σ, the relative entropy is UV-finite and manifestly computable,

$$S(\rho \,\|\, \sigma) = \operatorname{Tr}\big[C_\rho(\ln C_\rho - \ln C_\sigma)\big] + \operatorname{Tr}\big[(1 - C_\rho)(\ln(1 - C_\rho) - \ln(1 - C_\sigma))\big]. \tag{35}$$

Taking ρ the thermal state at inverse temperature β and σ the vacuum, both restricted to A, gives the two-interval thermal-to-vacuum relative entropy directly from the two correlators.

A controlled lattice evaluation (half-filled tight-binding chain, whose continuum limit is the $c = 1$ Dirac CFT — confirmed by the single-interval slope $dS/d\ln\ell = \frac{1}{3}$) yields two features (Fig. 5). First, the two-interval relative entropy exceeds the sum of its single-interval parts by a positive amount

$$\Delta S_{\rm rel} := S(\rho_{A_1A_2}\|\sigma_{A_1A_2}) - S(\rho_{A_1}\|\sigma_{A_1}) - S(\rho_{A_2}\|\sigma_{A_2}) > 0, \tag{36}$$

an inter-interval contribution carried entirely by the bilocal block of (34): it vanishes as the intervals separate ($\eta \to 0$) and grows as they merge ($\eta \to 1$), smoothly in the cross-ratio. Second, the mutual information is likewise a smooth function of η with *no* phase transition: unlike the holographic Proposition 9, the free fermion has $I(\eta) > 0$ for every $\eta > 0$ and no kink at $\eta = \frac{1}{2}$. This is the explicit, theory-dependent face of §7.6: the cross-ratio dependence is real and computable from the resolvent, but its analytic form — a smooth resolvent spectrum here, an RT surface exchange in the holographic case — is fixed by the operator content. The single-interval closed form (23) is universal; its two-interval continuation is not.

8 Further generalization: generalized qubits, qudits, and continuous variables

The construction so far concerns bipartite pure states of equal local dimension. We now record how far it extends. Three statements are rigorous (asymmetric dimension, spectral universality, and the Gaussian lift); one is a genuine boundary (mixed-state and multipartite entanglement).

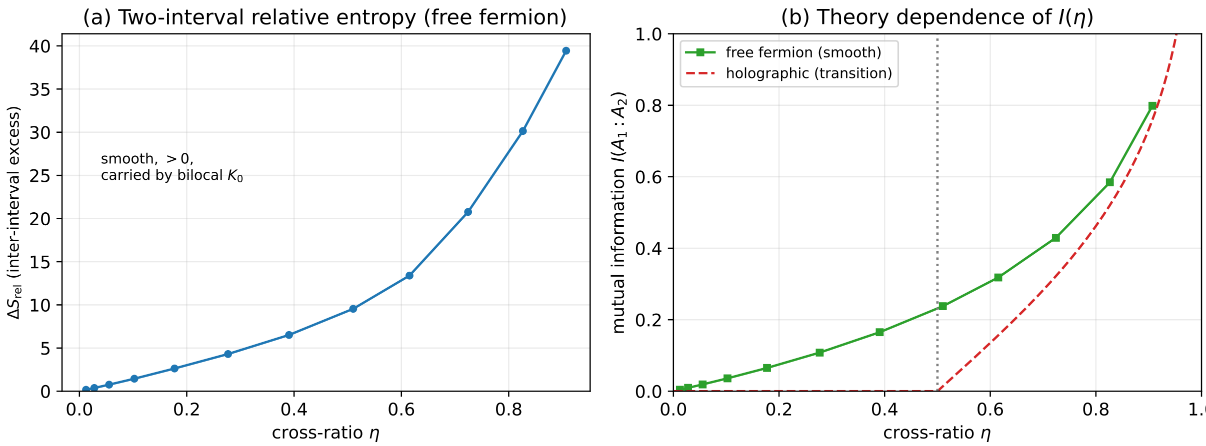

Figure 5: Free Dirac fermion ($c = 1$), two intervals, from the resolvent (34)–(35). **(a)** The inter-interval relative-entropy excess (36) of the thermal state to the vacuum, a positive function of the cross-ratio η carried by the bilocal modular Hamiltonian, vanishing as $\eta \to 0$. **(b)** The mutual information $I(A_1{:}A_2)$ versus η: smooth for the free fermion (no transition, nonzero for all $\eta > 0$), against the holographic result (33) with its Ryu–Takayanagi transition at $\eta = \frac{1}{2}$ — the theory dependence of §7.6 made quantitative.

8.1 Asymmetric local dimension is free

Proposition 10 (Only the Schmidt rank matters)**.** *For any pure state on $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$, the Schmidt decomposition has rank $n \le \min(d_A, d_B)$, and ρ_A, ρ_B share the nonzero spectrum $\{x_1, \ldots, x_n\}$. Every quantity in §2–4 (e_k, $\mathcal{C}_I$, the Maclaurin/Newton hierarchy, the moments p_k, the first-law deficit) depends only on this spectrum. Hence unequal "generalized qudit" dimensions $d_A \ne d_B$ add nothing: embedding a rank-n state in larger local spaces leaves all diagnostics invariant.*

8.2 Spectral universality and the mixed-state boundary

The symmetric-polynomial machinery is not special to entanglement; it is a statement about the spectrum of a density operator.

Proposition 11 (Spectral universality)**.** *For any density operator ρ on a finite-dimensional space, with eigenvalues $\{\lambda_i\}$, the elementary symmetric polynomials $e_k(\lambda)$, the moments $p_k = \operatorname{Tr} \rho^k$, Newton's identities (6), the generating function $\det(I + t\rho)$, the Maclaurin/Newton inequalities, and the Rényi entropies $S_\alpha = (1-\alpha)^{-1} \ln p_\alpha$ are all defined, and $h := \sqrt{e_2} = \sqrt{\frac{1}{2}(1 - \operatorname{Tr} \rho^2)}$ is a monotone function of the purity.*

The interpretation, however, is selective. For $\rho = \rho_A$ the reduction of a *pure* bipartite state, $h = \frac{1}{2}\mathcal{C}_I$ measures entanglement (Proposition 1). For a generic *mixed* operator, h measures *mixedness*, not entanglement.

Remark 2 (The mixed-state boundary)**.** For a mixed bipartite state ρ_{AB}, the I-concurrence is the convex roof $\mathcal{C}(\rho_{AB}) = \min_{\{p_i, \psi_i\}} \sum_i p_i \, \mathcal{C}(\psi_i)$ [RBC+01, Gou05], an optimization that is *not* a function of the single spectrum of ρ_A: separable mixed states can have ρ_A maximally mixed (so h maximal) yet zero entanglement. Thus Proposition 11 carries the bookkeeping but not the entanglement reading to mixed states; genuine mixed-state entanglement requires the convex roof. The first-law side (10), by contrast, is mixed-state native.

8.3 Continuous variables: the Gaussian lift via the symplectic spectrum

The oscillator thermofield double of §3 is the bridge to continuous-variable (CV) systems. A pure bipartite Gaussian state admits a phase-space Schmidt decomposition [BR03, ARL14] into independent two-mode-squeezed pairs with squeezing parameters $r_1, \dots, r_M$; the reduced state factorizes, $\rho_A = \bigotimes_{i=1}^{M} \rho_A^{(i)}$, with each factor a single-mode thermal state.

Proposition 12 (CV mode is the oscillator TFD)**.** *For a single two-mode-squeezed pair with squeezing r, the reduced Schmidt weights are $x_k = (1-q)\, q^k$ with $q = \tanh^2 r$ — the geometric distribution of the oscillator TFD (Proposition 3) under $q = e^{-\beta\epsilon} = \tanh^2 r$, i.e. mean occupation $\bar{n} = \sinh^2 r$. Consequently*

$$\operatorname{Tr}\big(\rho_A^{(i)}\big)^2 = \operatorname{sech} 2r_i, \qquad \mathcal{C}_I^{(i)} = \sqrt{2\big(1 - \operatorname{sech} 2r_i\big)}, \tag{37}$$

and for the multimode state the generating function and moments factorize over modes,

$$\det\big(I + t\rho_A\big) = \prod_{i=1}^{M} \det\big(I + t\rho_A^{(i)}\big), \qquad \operatorname{Tr}\rho_A^m = \prod_{i=1}^{M} \operatorname{Tr}\big(\rho_A^{(i)}\big)^m. \tag{38}$$

Proof. The two-mode-squeezed vacuum is $|r\rangle = \frac{1}{\cosh r} \sum_{k\geq 0} \tanh^k r\, |kk\rangle$, so $x_k = \tanh^{2k} r / \cosh^2 r = (1-q)q^k$ with $q = \tanh^2 r$ and $\operatorname{sech}^2 r = 1 - q$. Then $\operatorname{Tr}(\rho_A^{(i)})^2 = \frac{1-q}{1+q} = \operatorname{sech} 2r$ (Proposition 3, oscillator limit), and $\mathcal{C}_I$ follows from Proposition 1. Factorization is multiplicativity of det and of $\operatorname{Tr}(\cdot)^m$ over tensor products. □

Thus the entire symmetric-polynomial / moment apparatus applies to Gaussian states mode by mode, with the single-mode law $\operatorname{Tr}\rho^2 = \operatorname{sech} 2r$ replacing the qudit purity (8) (Fig. 6). As the squeezing or the number of modes grows, $\mathcal{C}_I \to \sqrt{2}$, the infinite-rank ceiling — the CV echo of the UV saturation in §5. Genuine CV entanglement is quantified by the logarithmic negativity / the symplectic eigenvalues [ARL14]; for pure Gaussian states this reduces to the per-mode entropies above.

8.4 Multipartite systems: a genuine boundary

For multipartite pure states there is no single Schmidt spectrum, and the construction extends only at the level of the one-party marginals $\{\rho_A, \rho_B, \rho_C, \dots\}$, each carrying its own symmetric-polynomial hierarchy (the marginal mixednesses). Genuine multipartite entanglement is not captured this way: it is measured by polynomial local-unitary/SLOCC invariants such as the three-tangle, a Cayley hyperdeterminant [CKW00], whose relation to symmetric functions of a single spectrum is lost. A multipartite "balance" theory would have to be built from the full invariant ring rather than from one spectrum, and is a separate undertaking.

9 Outlook

Sections 6–7 now settle both the single-interval reconstruction and the two-interval extension. The single-interval relative entropy (23) is closed form with explicit first law and positivity; its bulk symplectic-flux reconstruction is the rigid zeta series (29) (Theorem 2), every order beyond the first law being an RT-area variation (Proposition 8); the holographic two-interval relative entropy is the cross-ratio mutual information (33) with its phase transition; and the free-fermion two-interval thermal-to-vacuum relative entropy is computed from the resolvent (§7.7), smooth in the cross-ratio with the inter-interval excess (36) carried by the bilocal modular Hamiltonian. Three

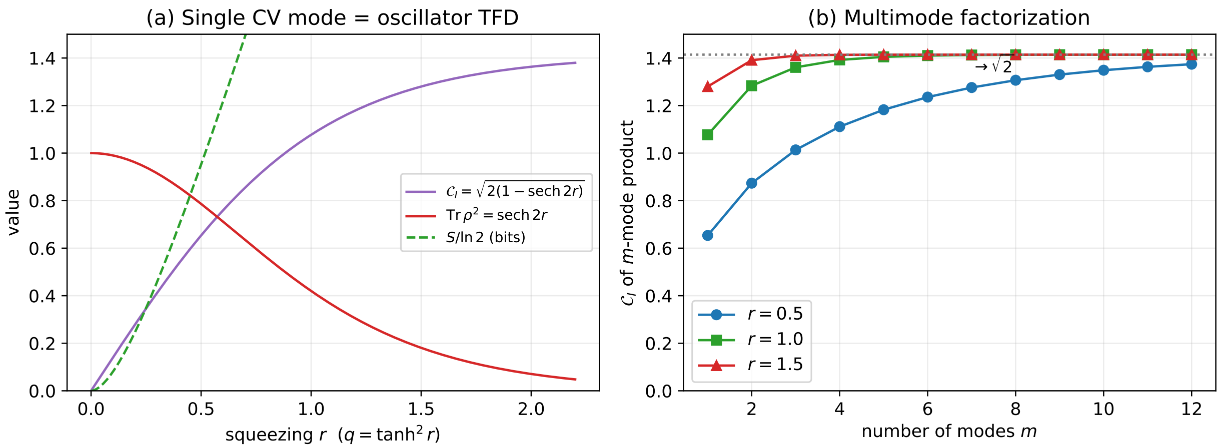

Figure 6: **(a)** A single continuous-variable mode (two-mode-squeezed pair) is the oscillator thermofield double with $q = \tanh^2 r$: purity $\mathrm{Tr}\,\rho^2 = \mathrm{sech}\,2r$, I-concurrence $\mathcal{C}_I = \sqrt{2(1 - \mathrm{sech}\,2r)}$, and entanglement entropy $S/\ln 2$ in bits, as functions of the squeezing r. **(b)** For an m-mode product of identical pairs the I-concurrence rises toward the infinite-rank ceiling $\sqrt{2}$, the continuous-variable counterpart of growing Schmidt rank.

continuations remain. First, evaluate the higher fluxes of Proposition 8 as genuinely independent symplectic integrals $\int_\Sigma \boldsymbol{\omega}$ in a theory with bulk matter, where the modular charge is no longer exactly linear and the area variations and symplectic fluxes separate. Second, extend the two-interval analysis to $N > 2$ intervals and to the thermal/holographic case with its multiple RT phases, where the modular Hamiltonian and the Fredholm determinant (14) acquire full multi-interval structure. Third, on the continuous-variable side (§8), the mode-wise reduction suggests a Gaussian analogue of the first-law comparison, with the symplectic eigenvalues playing the role of the Schmidt spectrum.

10 Conclusion

The two-qubit balance diagnostic generalizes cleanly to arbitrary Schmidt rank. Its altitude is the square root of the second elementary symmetric polynomial of the Schmidt spectrum and equals half the I-concurrence; its mean hierarchy is Maclaurin's inequality, with the concurrence bound as its first rung; and via Newton's identities it is interchangeable with the replica moments that compute entanglement entropy. The qudit and oscillator thermofield doubles provide exactly solvable realizations governed by the single purity formula (8), and the distinction between the maximal-entanglement point and the first-law point persists in every dimension with the closed-form offset (11), reducing to $\frac{1}{2}\tanh(\beta\epsilon/2)$ at $n = 2$, the simple qubit case. Lifted to the eternal BTZ black hole, the replica moments and the symmetric-polynomial generating function survive as a Fredholm determinant, while the concurrence saturates and loses discriminating power — so the geometric proximity measure is the relative entropy, not the balance deficit. Beyond qudits, the diagnostic is insensitive to asymmetric local dimension, its spectral machinery is universal (measuring mixedness off the pure-state locus), and it lifts to Gaussian continuous-variable systems mode by mode, each pair being the oscillator thermofield double; genuine mixed-state and multipartite entanglement lie outside the single-spectrum description and require, respectively, the convex roof and the hyperdeterminant invariants. Carried into an explicit bulk geometry, the relative-entropy comparison does become geometric: for a single interval in the planar-BTZ state it is the closed form $S(\rho \,\|\, \rho_0) = \frac{c}{3}[\frac{u^2}{6} - \ln(\sinh u/u)]$, whose first law is a cancellation of $O(u^2)$ terms, whose positivity is an elementary inequality on the zeros of sinh, and whose leading $\frac{c}{540}u^4$ is the quantum Fisher information, equal to the bulk canonical energy in the entanglement wedge — the simplex offset of Proposition 4 promoted to the gravitational symplectic form. Reconstructed from gravity, this relative entropy is the bulk functional $\Delta H_\xi - \text{Area}/4G$ on the full nonlinear geometry, whose symplectic-flux expansion is a rigid series in even zeta values ($\zeta(2)$ cancelled by the first law, $\zeta(4)$ the canonical energy) resumming to the Ryu–Takayanagi area; and for two intervals it becomes the holographic mutual information $\frac{c}{3}\ln[\eta/(1-\eta)]$, cross-ratio dependent with a phase transition at $\eta = \frac{1}{2}$, where the non-local modular Hamiltonian and the non-factorized Fredholm determinant take over from the single-interval closed form. Every flux order beyond the first law is a variation of the RT area, fixed rigidly by an even zeta value; and the two-interval relative entropy, computed for the free fermion from the resolvent, is smooth in the cross-ratio with a positive inter-interval excess carried by the bilocal modular Hamiltonian — the same measure that is universal for one interval becoming theory-dependent for two.

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A Newton's identities and the purity formula

Newton's identities. Taking the logarithmic derivative of the generating function (2),

$$\frac{d}{dt}\ln\prod_i(1+x_it)=\sum_i\frac{x_i}{1+x_it}=\sum_{k\geq 1}(-1)^{k-1}p_k\,t^{k-1},\qquad \frac{d}{dt}\ln\sum_j e_jt^j=\frac{\sum_j j\,e_jt^{j-1}}{\sum_j e_jt^j},$$

and matching powers of t yields (6). The low-order cases with $e_1=1$ read $p_2=1-2e_2$ and $p_3=1-3e_2+3e_3$, i.e. $e_2=\frac{1}{2}(1-p_2)$, $e_3=\frac{1}{6}(1-3p_2+2p_3)$.

Purity of the n-level TFD. With $x_k=q^k/Z_n$ and $Z_n=\sum_{k=0}^{n-1}q^k=\frac{1-q^n}{1-q}$,

$$\mathrm{Tr}\,\rho_R^2=\frac{1}{Z_n^2}\sum_{k=0}^{n-1}q^{2k}=\frac{(1-q)^2}{(1-q^n)^2}\cdot\frac{1-q^{2n}}{1-q^2}=\frac{(1-q)^2(1-q^n)(1+q^n)}{(1-q^n)^2(1-q)(1+q)}=\frac{(1-q)(1+q^n)}{(1+q)(1-q^n)}.$$

Setting $n=2$ gives $\frac{(1-q)(1+q^2)}{(1+q)(1-q^2)}=\frac{1+q^2}{(1+q)^2}$, and $n\to\infty$ (with $0<q<1$) gives $\frac{1-q}{1+q}$.